The weak interactions have many peculiar features that set them apart from QCD and E&M:
i) Every single matter particle in the SM exhibits weak interactions (only charged for E&M, only quarks for QCD)
ii) The force mediators are massive (unlike photons and gluons)
iii) The weak interactions violate parity, charge conjugation and CP.
iv) The weak interactions can change "flavor", i.e. particle type $\Rightarrow$ responsible for decays!

Perhaps the strangest part of the weak interactions is that they are not realized as a symmetry of the SM, at least not at room temperature type energies. We will eventually explain what this means and in fact this will solve the mediator mass issue by bringing in the Higgs's mechanism. But more on that later.

To keep in line with our development of E&M and QCD as theories of local (or gauge) invariance, we will go ahead and formulate the weak interactions in terms of a gauge symmetry. This is relevant since at some point in the history of the universe this is how it appeared. More on that later!

Perhaps the most surprising feature of formulating the weak interactions in terms of a gauge symmetry is that to do so, we are forced to "unify" the weak force with E&M!

Electroweak Unification

It is often said that the total gauge symmetry group of the SM is $\underbrace{SU(3)}^{QCD} \times \underbrace{SU(2)}^{Weak} \times \underbrace{U(1)}^{E+M}$

However this is <u>not</u> quite right. The correct groups are:

$$\overbrace{SU(3)}^{QCD} \times \overbrace{SU(2)_L \times U(1)_Y}^{\text{Electroweak}} \quad \text{High Energy}$$

$SU(2)_L$: left chiral; $U(1)_Y$: hypercharge

$$\Downarrow$$

$$\underbrace{SU(3)}_{QCD} \times \underbrace{U(1)_{EM}}_{E+M} \quad \text{Low Energy}$$

So we need to start with $SU(2)_L \times U(1)_Y$. The "L" in $SU(2)_L$ means that this symmetry is only relevant for "left-handed" fermion states. This is a bit of a misnomer since handedness has to do with helicity, whereas in actuality the $SU(2)$ acts on states of definite chirality (which does match helicity for massless particles).

Quick review of helicity vs. chirality. If we take an arbitrary spinor ψ, we can project it in 2 different ways:

$$\left.\begin{aligned} \tfrac{1}{2}(1 \pm S_{\vec{p}})\psi = \psi_{\pm}^{\text{helicity}} \quad &\Rightarrow \quad S_{\vec{p}}\,\psi_{\pm}^{h} = \pm \tfrac{\hbar}{2}\psi_{\pm}^{h} \\ \tfrac{1}{2}(1 \pm \gamma^5)\psi = \psi_{\pm}^{\text{chirality}} \quad &\Rightarrow \quad \gamma^5 \psi_{\pm}^{c} = \pm \psi_{\pm}^{c} \end{aligned}\right\}$$

and in general these eigenstates are different since $S_{\vec{p}} = S_z \propto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ while $\gamma^5 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

What's more we find that in general: helicity of a state is not Lorentz invariant, but in a given frame is constant
chirality of a state is Lorentz invariant, but in a given frame can change

Where this all becomes degenerate is when $m=0$, then chiral states are the same as helicity states.

Recall: $\mathcal{L}_{Dirac} = (\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$

$$= -\hbar c\left(i\psi_-^\dagger\partial_\mu\sigma^\mu\psi_- + i\psi_+^\dagger\partial_\mu\bar{\sigma}^\mu\psi_+\right) + mc^2(\psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_-)$$

This is in terms of 2-component objects ψ_+, ψ_- and σ's.

$\sigma^\mu = (I, \sigma^i)$ $\qquad$ $\bar{\sigma}^\mu = (I, -\sigma^i)$

To work w/ 4 component objects consider $\psi_R = \begin{pmatrix}\psi_+\\0\end{pmatrix}$ and $\psi_L = \begin{pmatrix}0\\\psi_-\end{pmatrix}$. Then $\psi = \begin{pmatrix}\psi_+\\\psi_-\end{pmatrix} = \psi_R + \psi_L$

But recall that $\qquad P_\pm = \frac{1}{2}(1 \pm \gamma^5) \Rightarrow P_+\psi = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\psi_R\\\psi_L\end{pmatrix} = \psi_R$, $P_-\psi = \psi_L$

However for $\bar{\psi} = i\psi^\dagger\gamma^0 \Rightarrow \overline{\psi_R} = \overline{\left(\frac{1+\gamma_5}{2}\psi\right)} = i\left(\frac{1+\gamma_5}{2}\psi\right)^\dagger\gamma^0 = i\psi^\dagger\frac{1+\gamma_5}{2}\gamma^0 = i\psi^\dagger\gamma^0\frac{1-\gamma_5}{2} = \bar{\psi}\frac{1-\gamma_5}{2}$

Then we can write: $\mathcal{L}_{Dirac} = (\hbar c)(\bar{\psi}_R + \bar{\psi}_L)\gamma^\mu\partial_\mu(\psi_R + \psi_L) + mc^2(\bar{\psi}_R + \bar{\psi}_L)(\psi_R + \psi_L)$

Note that: $\bar{\psi}_R\gamma^\mu\psi_L = \bar{\psi}\frac{1-\gamma_5}{2}\gamma^\mu\frac{1-\gamma_5}{2}\psi = \bar{\psi}\gamma^\mu\frac{1+\gamma_5}{2}\frac{1-\gamma_5}{2}\psi = \bar{\psi}\gamma^\mu\frac{1-\gamma_5^2}{2}\psi = 0$

$\Rightarrow$ All derivative terms mixing L w/ R vanish.

$\bar{\psi}_R\psi_R = \bar{\psi}\frac{1-\gamma_5}{2}\frac{1+\gamma_5}{2}\psi = 0 \Rightarrow$ All mass terms <u>not</u> mixing L w/ R vanish.

$$\mathcal{L}_{Dirac} = (\hbar c)\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + (\hbar c)\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + mc^2\bar{\psi}_R\psi_L + mc^2\bar{\psi}_L\psi_R$$

Everything here is in terms of 4-component objects.

$$\mathcal{L}_{Dirac} = (\hbar c)\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R + (\hbar c)\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + mc^2 \bar{\psi}_R \psi_L + mc^2 \bar{\psi}_L \psi_R$$

Now that we have L & R in the game, we need a "doublet" for the $SU(2)_L$ to act on.

Recall for quarks we introduced a triplet $\psi = \begin{pmatrix} \psi_R \\ \psi_b \\ \psi_g \end{pmatrix}$ for $SU(3)$.

We actually pair particles into left-handed doublets: $\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L$

And have right-handed singlets: $e_R, \mu_R, \tau_R, u_R, d_R, c_R, s_R, t_R, b_R$

What about $\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$? They don't exist! At least not for the massless neutrino story.

For simplicity we will focus on the first generation of leptons, i.e. $\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$, e_R and ignore the mass term (since it never plays a role in generating interactions).

$$\mathcal{L}_{Dirac} = (\hbar c)\bar{\chi}_L \gamma^\mu \partial_\mu \chi_L + (\hbar c)\bar{e}_R \gamma^\mu \partial_\mu e_R$$

$$\mathcal{L}_{Dirac} = (\hbar c)\,\bar{\chi}_L \gamma^\mu \partial_\mu \chi_L + (\hbar c)\,\bar{e}_R \gamma^\mu \partial_\mu e_R$$

This is invariant under global $\underbrace{SU(2)_L}_{\text{Isospin}} \times \underbrace{U(1)_Y}_{\text{Hypercharge}}$ where $SU(2)_L: \; e^{-ig\vec{\sigma}\cdot\vec{\theta}}\chi_L \qquad g = \frac{q}{\hbar c}$

$U(1)_Y: \; e^{-ig' Y_{\chi_L}\phi}\chi_L, \; e^{-ig' Y_{e_R}\phi} e_R$

With the factors Y_{χ_L} and Y_{e_R}, we allow χ_L and e_R to carry different amounts of the same hypercharge (governed by g').
Each of these can be promoted to a local symmetry of the Lagrangian by the methods we have covered in the preceding lectures.

So if we adjust g', then both Y_{χ_L} and Y_{e_R} scale accordingly!

Let: $\partial_\mu \chi_L \to D_\mu \chi_L = \partial_\mu \chi_L + ig\vec{\sigma}\cdot\underbrace{\vec{W}_\mu}_{\text{3 } SU(2)_L \text{ gauge fields}}\chi_L + ig' Y_{\chi_L} \overbrace{B_\mu}^{1\; U(1)_Y \text{ gauge field}} \chi_L$

$$\partial_\mu e_R \to D_\mu e_R = \partial_\mu e_R + ig' Y_{e_R} B_\mu e_R$$

Where: $\left.\begin{aligned} \vec{\sigma}\cdot\vec{W}'_\mu &= e^{-ig\vec{\sigma}\cdot\vec{\theta}}\,\vec{\sigma}\cdot\vec{W}_\mu\, e^{ig\vec{\sigma}\cdot\vec{\theta}} + \frac{i}{g}\partial_\mu\left(e^{-ig\vec{\sigma}\cdot\vec{\theta}}\right)e^{ig\vec{\sigma}\cdot\vec{\theta}} \\ B'_\mu &= B_\mu + \partial_\mu \phi \end{aligned}\right\}$ Gauge field transformation rules.

Then for each gauge field we introduce a kinetic term using $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ and again this would give the usual $\begin{cases} F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu & \text{for the } U(1)_Y \\ F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\underbrace{\epsilon^{abc}}_{SU(2)\text{ Lie algebra structure constants}} W^b_\mu W^c_\nu & \text{for } SU(2)_L \end{cases}$

The latter will include interactions between the $SU(2)_L$ gauge bosons (just like the gluons of QCD).

The $SU(2)_L$ interactions have a particularly interesting structure. We expect 3 gauge bosons $W^1_\mu, W^2_\mu, W^3_\mu$ which "act" on the left-handed doublets, e.g. $\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$.

Using the machinery of good old spin-$\frac{1}{2}$ from QM (that involved $SU(2)$ as well!) we can think of this in terms of:

$$W^3_\mu \sim \tfrac{1}{2}\sigma^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Rightarrow W^3_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\tfrac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad W^3_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$W^+_\mu \sim \tfrac{1}{2}\sigma^1 + i\tfrac{1}{2}\sigma^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow W^+_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad W^+_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$W^-_\mu \sim \tfrac{1}{2}\sigma^1 - i\tfrac{1}{2}\sigma^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow W^-_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad W^-_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Recall we use $\vec{\sigma}\cdot\vec{W}_\mu$ so then $\sigma^1 W^1_\mu + \sigma^2 W^2_\mu + \sigma^3 W^3_\mu$

These contain the action in flavor-space.

But notice that W^+_μ (W^-_μ) changes $e \to \nu_e$ ($\nu_e \to e$) respectively, so clearly the $\pm$ here must be connected to electric charge. These gauge bosons mediate interactions which change the electric charge of the matter involved.

The W^3_μ on the other hand does not change the particle type, hence does not change the electric charge, i.e. it is neutral.

There are 2 big problems:

i) We know that the weak gauge bosons are massive, and we already know that the Proca mass term $(\frac{mc}{\hbar})^2 W^a_\mu W^{\mu a}$ is not gauge invariant.

ii) Recall that to have mass terms for spinors requires both the L and R parts of ψ to combine, e.g. $mc^2 \bar{\psi}_L \psi_R$.
However we have just constructed a gauge theory where the L and R parts transform differently.
There is no way we can expect $mc^2 \bar{\psi}_L \psi_R$ to be gauge invariant!

Both of these problems will be solved w/ the Higgs mechanism for mass generation. A crucial part of this process is the breaking of $SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}$.

We will have much more to say about how such a symmetry can be broken, but for now we will just highlight the implication for the electroweak interactions.

$SU(2)_L \times U(1)_Y$ has 4 generators W^3_μ, $W^\pm_\mu$, B_μ. After symmetry breaking to $U(1)_{EM}$ we only expect one symmetry generator to survive. Which one is it?

You might have thought it would be B_μ, then $U(1)_Y \rightarrow U(1)_{EM}$, but that is <u>not</u> the case.

In actuality, the B_μ "mixes" with the neutral W^3_μ from $SU(2)_L$. We can form 2 orthogonal states:

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W \quad \Rightarrow \text{The photon of } U(1)_{EM}$$

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W \quad \Rightarrow \text{The massive neutral } Z^0 \text{ boson of the weak interactions.}$$

$\uparrow$

θ_W = Weinberg mixing angle

So it should be clear that we cannot identify just $U(1)_{EM}$ with the $U(1)$ factor in $SU(2)_L \times U(1)_Y$!

Going back to the original unified gauge group $SU(2)_L \times U(1)_Y$, it wouldn't make much sense to call this a "unified" group if the $SU(2)_L$ and $U(1)_Y$ factors had completely independent couplings g and g'. It turns out that they are in fact related. However since we experience the broken version of this theory, it is more useful to know how the couplings to $W^\pm$, Z^0 and γ are related.

Turns out: $g \sin\theta_W = g' \cos\theta_W = g_\gamma$ and $g = g_{W^\pm}$ while $g_Z = \frac{g_\gamma}{\sin\theta \cos\theta}$